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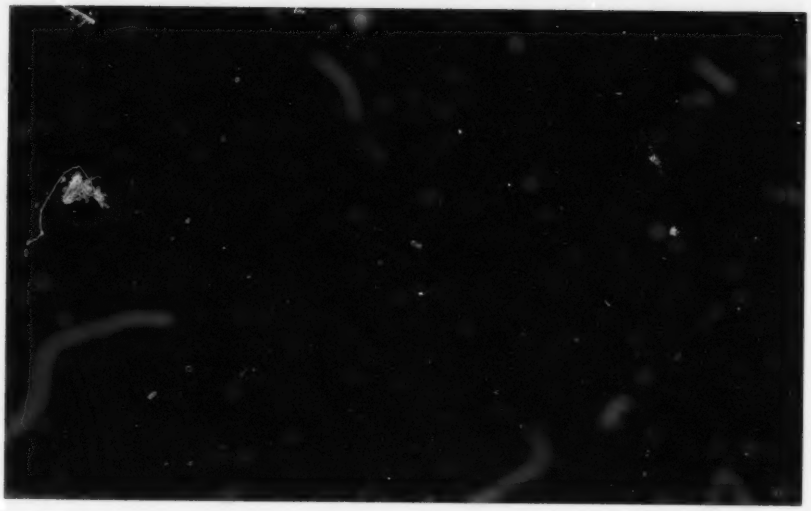

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A New Optimization Formulation of the Variational Inequality Problem with Application to the Network Equilibrium Problem

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ABSTRACT

In this paper, we develop a general differentiable equivalent optimization formulation for the variational inequality problem and analyse its properties. Descent methods are proposed for its solution and convergence proofs are given even in the case where the Jacobian is only positive semidefinite on the feasible set. The resulting algorithms are adapted for the network equilibrium problem in the space of path flows.

Keywords: variational inequalities; descent methods; optimization; network equilibrium; projection method; linearized Jacobi method.

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RÉSUMÉ

Dans cet article, nous développons une formulation d'un problème d'optimisation différentiable équivalente au problème d'inégalité variationnelle asymétrique et nous analysons ses propriétés. Nous proposons des méthodes de descente pour la solution de ce problème même quand le jacobien n'est seulement que positif semi-défini sur l'ensemble réalisable. L'algorithme qui en résulte est adapté pour le problème d'équilibre de réseau dans l'espace de flots de chemins.

Mots-clés: inégalité variationnelle; méthodes de descente; optimisation; équilibre de réseau; méthode de projection; méthode de Jacobi linéarisée.

1 Introduction.

The variational inequality problem we study, VIP, is defined as follows: find $v \in \Omega_v$ such that

$$VIP_v \qquad s(v)^T(u-v) \geq 0, \quad \text{for all} \quad u \in \Omega_v \tag{1}$$

or alternatively let

$$v = \delta h, \qquad S(h) = \delta^T s(v) \tag{2}$$

and find $h \in \Omega_h$ such that

$$VIP_h \qquad s(\delta h)^T \delta(u-h) \geq 0, \quad \text{for all} \quad u \in \Omega_h \tag{3}$$

where

$\Omega_v \subset R^m$ and $\Omega_h \subset R^n$ are both convex and compact sets;
$v \in R^m$ and $h \in R^n$ are both vectors;
$s(v) : \Omega_v \subset R^m \rightarrow R^m$ is a mapping;
$S(h) : \Omega_h \subset R^n \rightarrow R^n$ is a mapping;
δ is an $m \times n$ linear transformation matrix;
^{T} denotes transposition.

It is well known that when $\nabla s(v)$ is symmetric and positive semidefinite on Ω_v, VIP_v corresponds to the optimality condition of a convex program; if $\nabla s(v)$ does not satisfy the condition, then there is in general no such convex program.

Both VIP_v and VIP_h have been widely used to model network equilibrium problems arising in transportation system analysis (Smith, 1979, 1983, Dafermos, 1980, Bertsekas and Gafni,1982, Florian and Spiess, 1984, Florian, 1986, Marcotte and Guélat, 1988) and regional science (Florian and Los, 1982, Nagurney, 1987, Dafermos and Nagurney, 1987). The study of variational inequalities draws on nonlinear optimization methodology, solution of nonlinear equations and fixed point theory.

In order to solve VIP_v or VIP_h, two main approaches have emerged: the iterative approach, based on solving a sequence of simpler problems (optimization problems, linear or nonlinear complementarity problems, such as Pang and Chan, 1982), and the optimization approach, where the variational inequality is solved as a mathematical program (either differentiable and nondifferentiable, such as Nguyen and Dupuis, 1984, Marcotte, 1985,

Marcotte and Dussault, 1990). The first approach draws on the theory of the iterative solution of nonlinear systems of equations (Ortega and Rheinboldt, 1970) and includes algorithms based on SOR, Gauss-Seidel, Newton, linearized Jacobi and projection methods. Among others, Bertsekas and Gafni (1982) present a projection method for VIP_h and Dafermos (1982, 1983) presents a general iterative scheme including the projection method for VIP_v as a special case. Nguyen and Dupuis (1984) propose a cutting plane approach for VIP_v. Marcotte (1985) proposes an algorithm for the solution of an equivalent nondifferentiable optimization formulation of VIP_v for monotone $s(v)$, while Marcotte and Dussault (1990) solve an equivalent optimization formulation for VIP_v by solving a sequence of linear programs. Fukushima (1986) proposes a modified projection method for VIP_v, where each iteration requires a projection onto a supporting half space, rather than just the solution space itself. Harker (1988) develops a heuristic acceleration method for the Jacobi and projection algorithms for solving monotone VIP_v which is reminiscent of the PARTAN variant for primal optimization algorithms. Pang (1990) studies the Newton method for solving a system of B-differentiable equations and applies the resulting method to the nonlinear complementarity, variational inequality and nonlinear programming problems. Harker and Pang (1990) provide for a comprehensive survey of the subject.

A significant contribution to the reformulation of VIP_v as an equivalent differentiable optimization problem is that of Fukushima (1989) who develops a continuously differentiable and pseudoconvex equivalent optimization formulation for VIP_v when $\nabla s(v)$ is asymmetric, by using the projection operator. We shall refer to this formulation as the projection based optimization model, $PBOM$.

Motivated by Fukushima's work, we consider a reformulation which we refer to as the linearized Jacobi based optimization model, $LJBOM$, for VIP_h. A prevailing difficulty particular to VIP_h is that the mapping $\nabla S(h) = \delta^T \nabla s(v) \delta$ may not be positive definite and the convergence of several classical methods in this case is not guaranteed. In particular, for the network equilibrium problem in the space of path flows, $\delta^T \delta$ is singular. We extend Fukushima's theory to demonstrate that even when $\delta^T \delta$ is singular, our proposed method provides a descent direction and is globally convergent. In particular, our method can be implemented more easily for VIP_h than for VIP_v in the case of the network equilibrium problem.

This paper is organized as follows. In section 2, the required notations and definitions are introduced. Next, $LJBOM$ is formulated and its struc-

ture is analyzed. Section 4 deals with a class of descent methods and a proof of convergence for such algorithms. In section 5, the solution approach is adapted to the network equilibrium problem in the space of path flows. Finally, section 6 concludes the paper and proposes certain directions for future research.

2 Definitions and assumptions.

Let the B-norm in R^n be defined by $||x||_B = (x^T B x)^{1/2}$ where B is a symmetric positive definite matrix. Given the norm $\|\cdot\|_B$, the projection of a point x onto the closed convex set Ω, denoted by $\mathrm{Proj}_{\Omega,B(x)}$, is defined as the unique solution of the following quadratic program:

$$\min_{y \in \Omega} \|y - x\|_B. \tag{4}$$

The projection operator $\mathrm{Proj}_{\Omega,B}(\cdot)$ is nonexpansive, i.e.,

$$\|\mathrm{Proj}_{\Omega,B}(x) - \mathrm{Proj}_{\Omega,B}(y)\|_B \leq \|x - y\|_B, \quad \text{for all} \quad x, y \in R^n. \tag{5}$$

Moreover, for each $x \in R^n$, $\mathrm{Proj}_{\Omega,B}(x)$ satisfies

$$(x - \mathrm{Proj}_{\Omega,B}(x))^T B(y - \mathrm{Proj}_{\Omega,B}(x)) \leq 0 \quad \text{for all} \quad y \in \Omega. \tag{6}$$

The proofs of these results for the special case where $B = I$ may be found in Auslender (1976, chapter 5) and Kinderlehrer and Stampacchia (1980, chapter 1), and can be extended to the B norm in a straightforward manner. The mapping $S(x) : R^n \to R^n$ is monotone on Ω if

$$(S(x) - S(y))^T (x - y) \geq 0 \quad \text{for all } x, y \in \Omega, \tag{7}$$

strictly monotone on Ω if strict inequality holds in (7) whenever $x \neq y$, and strongly monotone on Ω if there is a λ positive such that

$$(S(x) - S(y))^T (x - y) \geq \lambda \|x - y\|_2^2 \quad \text{for all } x, y \in \Omega. \tag{8}$$

It is well known (Ortega and Rheinboldt, 1970) that, if S is continuously differentiable and the Jacobian matrix $\nabla S(x)$ is positive definite on Ω, i.e.,

$$(x-y)^T \nabla S(x)(x-y) \geq \lambda \|x-y\|_2^2 \quad \text{for all } x, y \in \Omega, \quad x \neq y \in R^n, \tag{9}$$

then S is strongly monotone on Ω with λ the minimum eigenvalue of the symmetric part of $\nabla S(x)$ on Ω. Note that $\nabla S(x)$ need not be symmetric in (9).

Finally we recall that a mapping $S(x) : R^n \to R^n$ is Lipschitz continuous on Ω if there exists a constant L_s such that

$$\|S(x) - S(y)\| \leq L_s \|x - y\| \quad \text{for all } x, y \in \Omega. \tag{10}$$

3 Equivalent formulations of VIP_h.

We consider first the following basic propositions. First various equivalent formulations for VIP_h are proposed and characterized.

Proposition 3.1 *Let $B(h)$ be an $n \times n$ symmetric, continuous and positive definite matrix for all $h \in R^n$ and α a positive parameter. The following three problems are equivalent for a given h:*

$$(i) \quad \min F(h,u) = S(h)^T(u-h) + \frac{1}{2\alpha}(u-h)^T B(h)(u-h), \quad u \in \Omega_h \tag{11}$$

$$(ii) \quad \min \|u - (h - \alpha B^{-1}(h)S(h))\|^2_{B(h)}, \quad u \in \Omega_h \tag{12}$$

$$(iii) \quad H(h) = Proj_{\Omega_h, B(h)}(h - \alpha B(h)^{-1} S(h)). \tag{13}$$

Proposition 3.2 (Bertsekas and Gafni, 1982) *Assume*

$$\tilde{h}(h) = \arg\min\{\|h - y\|_{\overline{B}} \mid y \in H^*\}$$

where $H^* = \{h | v^* = \delta h\}$, v^* is a unique solution of VIP_v, $\overline{B}$ is a symmetric positive definite matrix and

$$\overline{h}(h) = \arg\min\{\|h - y\|_{\overline{B}} | y \in H^* \cap \Omega_h\}. \tag{14}$$

Then there exists a positive scalar η such that for all $h \in \Omega$,

$$\|h - \tilde{h}(h)\|_{\overline{B}} \geq \eta \|h - \overline{h}(h)\|_{\overline{B}}. \tag{15}$$

Proposition 3.3 *Suppose that there exist a symmetric positive definite matrix $\overline{B}$ with $B(h) - \overline{B}$ positive semi-definite, for all $h \in \Omega_h$, and a parameter $\theta, 0 < \theta < 1$ such that*

$$\|\overline{B}^{-1}\{\alpha[S(\overline{h}(h)) - S(h)] - B(h)[\overline{h}(h) - h]\}\|_{\overline{B}} \leq \theta \|\overline{h}(h) - h\|_{\overline{B}} \tag{16}$$

holds for all $h \in \Omega_h$, where $H^* = \{h | v^* = \delta h\}$ and v^* is a unique solution of VIP_v. Then h solves the variational inequality VIP_h if and only if h is a fixed point of the mapping H, i.e., $h = H(h)$.

Proof. This is a direct consequence of Theorem 3.2.1 in the paper by Wu, Florian and Marcotte (1991). □

Proposition 3.4 (Wu, Florian and Marcotte, 1991, Proposition 3.2.3) *Suppose that there exists a symmetric positive definite matrix $\overline{B}$ with $B(h) - \overline{B}$ positive semidefinite for all $h \in \Omega_h$. The requirement (16) is satisfied if $s(v)$ is Lipschitzian with Lipschitz constant L_s, strongly monotone on Ω with strong monotonicity constant λ, and the following condition holds:*

$$1 > \theta^2 \geq \omega - 2\alpha\lambda\gamma\eta + \alpha^2 L_s \beta^2 > 0 \tag{17}$$

where

α is a positive parameter;
Λ_{max} is the maximum eigenvalue of $\overline{B}$;
Λ_{min} is the minimum eigenvalue of $\overline{B}$;
$\beta = \|\delta\|_F^2 \Lambda_{min}^{-1}$;
$\gamma = \|\delta\|_F^2 \Lambda_{max}^{-1}$;
ω is a positive parameter larger than 1;
η is given in (15).

With these propositions, we are ready to study a new formulation which is inspired from Fukushima's work (1989). Consider the problem:

$$LJBOM \qquad \min_{h \in \Omega} f(h) \tag{18}$$

where

$$f(h) = -S(h)^T(H(h) - h) - \frac{1}{2\alpha}(H(h) - h)^T B(h)(H(h) - h) \tag{19}$$

$$= -\min_{u \in \Omega_h} F(h, u). \tag{20}$$

Theorem 3.1 *Let the function $f : R^n \to R^1$ be defined by (19) and let (16) hold. Then $f(h) \geq 0$ for $h \in \Omega_h$ and $f(h) = 0$, i.e., h solves $LJBOM$ in (18) if and only if h solves VIP_h.*

Proof. For the first part, since $f(h)$ is the negative of the optimal value of problem (11) which is less or equal to zero due to $h \in \Omega_h$, it is clear that $f(h) \geq 0$ for every $h \in \Omega_h$.

For the second part, suppose that h solves VIP_h. Then, by Proposition 3.3, we have $h = H(h)$, which along with (19) implies $f(h) = 0$. Conversely suppose that $f(h) = 0$. It then follows from (19) that

$$-S(h)^T(H(h) - h) = \frac{1}{2\alpha}(H(h) - h)^T B(h)(H(h) - h). \tag{21}$$

However, since $H(h) = \text{Proj}_{\Omega_h, B(h)}(h - \alpha B(h)^{-1} S(h))$, it follows that

$$(h - \alpha B(h)^{-1} S(h) - H(h))^T B(h)(u - H(h)) \leq 0 \quad \text{for all} \quad u \in \Omega_h. \tag{22}$$

Putting $u = h$ in (21) and rearranging terms, we obtain

$$S(h)^T(h - H(h)) \geq \frac{1}{\alpha}(H(h) - h)^T B(h)(H(h) - h)). \tag{23}$$

Because $B(h)$ is positive definite, considering (21) and (23), it follows that $h = H(h)$. Consequently, by Proposition 3.3, h is a solution of VIP_h. □

Note that $LJBOM$ reduces to $PBOM$ if $B(h) = \overline{B}$, where $\overline{B}$ is a symmetric positive definite matrix. $LJBOM$ for VIP_v is Fukushima's model when $B(h) = \overline{B}$. This highlights one of the differences between his approach and ours.

In what follows, we discuss some important characteristics of $LJBOM$. First we note that f is a differentiable function.

Proposition 3.5 *If the mapping $s(v) : \Omega_v \subset R^m \to R^m$ is continuous, then the function $f(h) : \Omega_h \subset R^n \to R^1$ defined by (19) is also continuous. Furthermore, if $s(v)$ is continuously differentiable, then f is continuously differentiable and its gradient is given by*

$$\begin{aligned} \nabla f(h) &= -\nabla_h F(h, H(h)) \\ &= S(h) - (\nabla S(h) - \frac{B(h)}{\alpha})^T (H(h) - h) \\ &\quad -\frac{1}{2\alpha}(H(h) - h)^T \nabla B(h)(H(h) - h) \end{aligned} \tag{24}$$

where $F(h, u)$ is defined in (11), and $\nabla B(h)$ in $R^n \times R^n \times R^n$ is the derivative of the matrix $B(h)$ in $R^n \times R^n$.

Proof. If $s(v) : \Omega_v \subset R^m \to R^m$ is continuous, then it follows from the (2), (13) that the mappings $S(h)$ and $H(h)$ are continuous. Hence, by (19), the function f is also continuous. For the second half of the proposition, we know that F is continuous. Since, by definition,

$$f(h) = -\min\{F(h, u) | u \in \Omega_h\} \tag{25}$$

and the minimum of the right-hand side of (25) is uniquely obtained at $u = H(h)$, it follows from Auslender (1976, ch. 4, Theorem 1.7) that f is differentiable and $\nabla f(h)$ is given by (24). □

Corollary 3.1 *If h is optimal, then $\nabla f(h) = S(h)$.*

Proof. By (24), we can see that if h is optimal, then $h = H(h)$ and therefore the result holds. □

Proposition 3.6 *Suppose that (16) and (14) are satisfied. Then*

$$\|H(h) - \overline{h}(h)\|_{B(h)} \leq \theta \|h - \overline{h}(h)\|_{B(h)}. \tag{26}$$

Proof. From the definition of $H(h)$ and $\overline{h}(h)$, we have

$$[S(h) + \frac{B(h)}{\alpha}(H(h) - h)]^T [\overline{h}(h) - H(h)] \geq 0$$

and

$$S(\overline{h}(h))^T [H(h) - \overline{h}(h)] \geq 0.$$

Adding the two inequalities, we get

$$\begin{aligned} &[H(h) - \overline{h}(h)]^T \cdot B(h)[H(h) - \overline{h}(h)] \\ &\quad \leq [H(h) - \overline{h}(h)]^T \{\alpha[S(\overline{h}(h)) - S(h)] - B(h)[\overline{h}(h) - h]\}. \end{aligned}$$

Thus:

$$\begin{aligned} \|H(h) - \overline{h}(h)\|^2_{B(h)} &\leq \|H(h) - \overline{h}(h)\|_{B(h)} \\ &\quad \cdot \|B(h)^{-1}\{\alpha[S(\overline{h}(h)) - S(h)] - B(h)[\overline{h}(h) - h]\}\|_{B(h)}. \end{aligned}$$

According to (16), we have

$$\begin{aligned} &\|B(h)^{-1}\{\alpha[S(\overline{h}(h)) - S(h)] - B(h)[\overline{h}(h) - h]\}\|_{B(h)} \\ &= \|\overline{B}^{-1}\{\alpha[S(\overline{h}(h)) - S(h)] - B(h)[\overline{h}(h) - h]\}\|_{\overline{B}} \\ &\leq \theta \|\overline{h}(h) - h\|_{\overline{B}} \\ &\leq \theta \|\overline{h}(h) - h\|_{B(h)}. \end{aligned}$$

Therefore we obtain $\|H(h) - \overline{h}(h)\|^2_{B(h)} \leq \theta \|H(h) - \overline{h}(h)\|_{B(h)} \|\overline{h}(h) - h\|_{B(h)}$. Dividing by $\|H(h) - \overline{h}(h)\|_{B(h)}$, there comes the result. □

Proposition 3.7 *Denote $v = \delta h$ and $v_H = \delta H(h)$. If the condition (16) is satisfied, then δ is a one-to-one and onto mapping that takes $H(h) - h$ into $v_H - v$.*

Proof. By using (26), we prove the proposition by contradiction. Suppose that we have $H(h) \neq h$ and $v_H = v$. Then

$$v_H - v^* = \delta(H(h) - \overline{h}(h)) \tag{27}$$

and

$$v - v^* = \delta(h - \overline{h}(h)). \tag{28}$$

Substituting (27) from (28) and rearranging, we have

$$\delta(H(h) - \overline{h}(h)) = \delta(h - \overline{h}(h)).$$

which is equivalent to

$$\|H(h) - \overline{h}(h)\|_{B(h)} = \|h - \overline{h}(h)\|_{B(h)}.$$

This is in contradiction with (26). That is, if there is $H(h) \neq h$, then there must exist a $v_H \neq v$. □

Proposition 3.8 $\|v_H - v\|_2 \geq \|\delta^+\|_F^{-1}\|H(h) - h\|_2$ *where* $v_H = \delta H(h)$, δ^+ *is the Moore-Penrose inverse of* δ *and* $\|\delta^+\|_F$ *the Frobenius norm of the matrix* δ^+.

Proof. Suppose $v_H - v = \delta(H(h) - h)$. According to Proposition 3.7, δ is invertible, i.e., one-to-one and onto. Therefore by Wu, Florian and Marcotte (1991, Proposition A.1), we have the result. □

Proposition 3.9 *Assume that the mapping* $s(v) : \Omega_v \subset R^m \to R^m$ *is continuously differentiable, that its Jacobian matrix* $\nabla s(v)$ *is positive definite and* $\nabla B(h)(H(h) - h)$ *is positive semidefinite. If* h *is a stationary point for problem (18), i.e.,*

$$\nabla f(h)(u - h) \geq 0 \quad \text{for all} \quad u \in \Omega_h, \tag{29}$$

then h *is a global optimal solution of problem (18) and hence it solves the variational inequality (3).*

Proof. Since $\nabla s(v)$ is positive definite, then

$$\nabla S(h) = \delta^T \nabla s(v) \delta \tag{30}$$

is clearly positive semidefinite. However, due to Proposition 3.7, we know that

$$\begin{aligned}(H(h)-h)^T\nabla S(h)(H(h)-h) &= (v_H-v)^T\nabla s(v)(v_H-v)\\ &> 0 \quad \text{if } H(h)\neq h,\end{aligned}$$

where $v_H = \delta H(h)$ and $v = \delta h$. Suppose that h satisfies (29). Substituting (24) into (29) and letting $u = H(h)$, we have

$$\begin{aligned}&(S(h)-(\nabla S(h)-\frac{B(h)}{\alpha})^T(H(h)-h)\\ &\quad -\frac{1}{2\alpha}(H(h)-h)^T\nabla B(h)(H(h)-h))^T(H(h)-h)\geq 0 \qquad (31)\end{aligned}$$

and

$$\begin{aligned}&(S(h)+\frac{B(h)}{\alpha}(H(h)-h))^T(H(h)-h)\geq\\ &\quad (H(h)-h)^T\nabla S(h)(H(h)-h)+\\ &\quad \frac{1}{2\alpha}((H(h)-h)^T\nabla B(h)(H(h)-h))^T(H(h)-h). \qquad (32)\end{aligned}$$

On the other hand, since $H(h) = \mathrm{Proj}_{\Omega_h,B(h)}(h-\alpha B(h)^{-1}S(h))$, it follows from

$$(h-\mathrm{Proj}_{\Omega_h,B(h)}(h))^T B(h)(u-\mathrm{Proj}_{\Omega_h,B(h)}(h))\leq 0 \quad \text{for all } u\in\Omega_h \qquad (33)$$

that

$$(h-\alpha B(h)^{-1}S(h)-H(h))^T\frac{B(h)}{\alpha}(u-H(h))\leq 0 \quad \text{for all } u\in\Omega_h. \qquad (34)$$

In particular, putting $u = h$ in (34), we obtain

$$(S(h)+\frac{B(h)}{\alpha}(H(h)-h))^T(H(h)-h)\leq 0. \qquad (35)$$

Thus by (32) and (35), we have

$$\begin{aligned}&(H(h)-h)^T\nabla S(h)(H(h)-h)\\ &\quad +\frac{1}{2\alpha}((H(h)-h)^T\nabla B(h)(H(h)-h))^T(H(h)-h)\leq 0 \qquad (36)\end{aligned}$$

which can only be satisfied if $h = H(h)$. By Proposition 3.3, this implies that h solves VIP_h and by Theorem 3.1, h is a global optimal solution of problem (18). □

An important consequence of this proposition is that any local optimal solution of (18) is also a global optimal one of (18) and vice versa, which means that $f(h)$ is at least a pseudo-convex function. Thus any algorithm that finds a locally optimal solution can be used to solve (18).

Proposition 3.10 *Let h be a solution of (18) and suppose that h is an interior point of Ω_h. Then $S(h) = 0$.*

Proof. See (Kinderlehrer and Stampacchia, 1980, ch.1, Corollary 3.2) and Theorem 3.1. □

Proposition 3.11 *Let h be a solution of (18). Then $S(h)$ determines a hyperplane of support for Ω_h provided $S(h) \neq 0$.*

Proof. See (Kinderlehrer and Stampacchia, 1980, ch.1, Corollary 3.4) and Theorem 3.1. □

Next, we present several equivalent formulations of (18) which are useful in understanding the structure of VIP. The following equivalences do not require monotonicity on the convex set Ω_v.

Proposition 3.12 *The formulation given in (18) is equivalent to the formulation:*

$$\max_{h \in \Omega_h} \min_{u \in \Omega_h} S(h)^T(u-h) + \frac{1}{2\alpha}(u-h)^T B(h)(u-h), \tag{37}$$

itself equivalent to the bilevel formulation

$$\min_{h \in \Omega_h} -S(h)^T(u-h) - \frac{1}{2\alpha}(u-h)^T B(h)(u-h)$$
$$u \in \arg\min_{u \in \Omega_h} S(h)^T(u-h) + \frac{1}{2\alpha}(u-h)^T B(h)(u-h). \tag{38}$$

Proof. Straightforward from the definition of $LJBOM$ and of a bilevel program (Bard and Falk, 1982) □

4 Descent methods

Theorem 4.1 *Let the mapping $s(v) : \Omega_v \subset R^m \to R^m$ be continuously differentiable. If the condition (16) holds, the Jacobian matrix $\nabla s(v)$ and $\nabla B(h)(H(h)-h)$ are both positive definite on Ω_v and Ω_h respectively. Thus for each $h \in \Omega_h$ which is not solution to VIP_h, the vector $d = H(h) - h$ satisfies the descent property*

$$\nabla f(h)^T d < 0$$

for every $d \neq 0$. Moreover

$$\nabla f(h)^T d \leq -(\mu(\|\delta^+\|_F^{-1})^2 + \frac{\eta}{2\alpha})\|d\|_2^2$$

where $0 < \mu \leq \|\nabla s(v)\|$ and $0 < \eta \leq \|\nabla B(h)(H(h)-h)\|$.

Proof. From (24), we have

$$\begin{aligned}\nabla f(h)^T d = &(S(h) + \frac{B(h)}{\alpha}(H(h)-h))^T(H(h)-h)\\ &-(H(h)-h)^T\nabla S(h)(H(h)-h)\\ &-\frac{1}{2\alpha}((H(h)-h)\nabla B(h)(H(h)-h))^T(H(h)-h). \qquad (39)\end{aligned}$$

Since

$$(h - \alpha B(h)^{-1}S(h) - H(h))^T B(h)(u - H(h)) \leq 0 \quad \text{for all} \quad u \in \Omega_h,$$

we obtain, after setting $u = h$ in the above inequality:

$$(S(h) + \frac{B(h)}{\alpha}(H(h)-h))^T(H(h)-h) \leq 0. \qquad (40)$$

Combining (39), (40) and Proposition 3.8, there comes

$$\begin{aligned}\nabla f(h)^T d &\leq -d^T\nabla S(h)d - \frac{1}{2\alpha}d^T\nabla B(h)(H(h)-h)d\\ &= -d^T\delta^T\nabla s(v)\delta d - \frac{1}{2\alpha}d^T\nabla B(h)(H(h)-h)d\\ &\leq -(v_H - v)^T\nabla s(v)(v_H - v) - \frac{\eta}{2\alpha}d^T d\\ &\leq -\mu\|v_H - v)\|_2^2 - \frac{\eta}{2\alpha}\|d\|_2^2\\ &\leq -(\mu(\|\delta^+\|_F^{-1})^2 + \frac{\eta}{2\alpha})\|d\|_2^2.\end{aligned}$$

□

Corollary 4.1 *For PBOM, if* $\|\nabla s(v)\| \geq \mu > 0$, *then* d *is a descent direction, i.e.,*

$$\nabla f(h)^T d < -\mu(\|\delta^+\|_F^{-1})^2 \|d\|_2^2.$$

Proof. This is a direct consequence of Theorem 4.1. □

Theorem 4.1 and Corollary 4.1 are interesting because the descent direction is still guaranteed even though $\nabla S(h)$ is positive semidefinite. In the following, we present an algorithmic framework using the descent direction d.

Theorem 4.2 *Let* $\{h^l\}$ *be a sequence generated by the iteration*

$$f^{l+1} = h^l + t_l d^l, \qquad l = 0, 1, 2, \ldots \tag{41}$$

where $d^l = H(h^l) - h^l$ *and* $t_l \in [0, 1]$ *is determined from*

$$f(h^l + t_l d^l) = \min\{f(h^l + t d^l) | \ \ 0 \leq t \leq 1\} \tag{42}$$

Let Ω_h *be compact and convex, the mapping* $s(v) : \Omega_v \subset R^m \to R^m$ *be continuously differentiable ,* $\nabla s(v)$ *be positive definite on* Ω_v *and* $\nabla B(h)(H(h) - h)$ *be positive semidefinite for all* $h \in \Omega_h$ *and let the condition (16) hold. Then, for any starting point* $h^0 \in \Omega_h$, *the generated sequence* $\{h^l\}$ *lies in* Ω_h *and converges to a solution of* VIP_h.

Proof. Since the points h^l and $H(h^l) = h^l + d^l$ both belong to Ω_h and since $0 \leq t_l \leq 1$, it follows, from the convexity of Ω_h, that the sequence $\{h^l\}$ is contained in Ω_h. By compactness of Ω_h, all limit points of the sequence $\{h^l\}$ stay in Ω_h.

To prove the convergence of the algorithm, we use Zangwill's Convergence Theorem A (Zangwill, 1969). The assumption that Ω_h is compact ensures that Condition 1 of Convergence Theorem A holds. Condition 2 (the descent property) is ensured also by Theorem 4.1. Now consider Condition 3, i.e., that the algorithmic map is closed. The algorithmic map which is a point-to-set map is defined as

$$A(h^l) = M(h^l)D(h^l)$$

where $M(h^l)$ is the search map and $D(h^l)$ is the direction map, i.e., $D(h^l) = (h^l, d^l) = (h^l, H(h^l) - h^l)$. Since $d^l = H(h^l) - h^l$ is continuous with respect of h^l, whenever $s(v)$ is continuous, $D(h^l)$ is continuous. It is well known that the search algorithm determined by (42) induces the continuity of $M(h)$. In

this case, $A(h)$ is also continuous. This concludes that the algorithmic map is continuous, hence closed, and $f(h^l) \to 0$. This implies, by Theorem 3.1, that $\{h^l\}$ converges a solution of VIP_h and any subsequence converges to the solution. □

The proposed algorithm is difficult to implement due mainly to the difficulties of obtaining h in closed form. Therefore it is in general computationally impossible to carry out (42) exactly.

Fukushima (1989) adopted the inexact Armijo-type steplength rule to solve (42) without specifying the selection of the parameters in the method. Our solution method is basically the same, but it exploits the properties of the function f. This function has only one "valley". Suppose we are at one side of the "valley". If we find that the objective increases at a given steplength, we try to reduce it by reducing the steplength by half and evaluate the objective again to see if it is reduced. If not, the steplength is reduced by half again. Specifically, for the current iteration, with h^l and d^l, the steplength t_l is determined as $t_l = (\frac{1}{2})^k$, where k is the smallest nonnegative integer such that

$$f(h^l + t_l d^l) < f(h^l).$$

We note that if t is set to be equal to 1, then the algorithm corresponds to the projection method or the linearized Jacobian method, depending on whether B is constant or varies with h. The algorithm, however, is not of the descent type in general.

5 The network equilibrium problem.

It is well known that the network equilibrium problem may be formulated as VIP_v (Smith, 1979, 1983, and Dafermos, 1980) and VIP_h (Bertsekas and Gafni, 1982). We consider the problem in the path flow space, which is equivalent to VIP_h, i.e., find $h^* \in \Omega_h$, an equilibrium path flow, such that

$$S(h^*)(h - h^*) \geq 0, \qquad \text{for all} \quad h \in \Omega_h$$

where:

h^* is an equilibrium path flow vector in R^n;
h is any feasible path flow vector in R^n;
$S(h^*)$ is the path cost vector in R^n corresponding to h^*;

Ω_h is a feasible region defined as follows:

$$\sum_{k \in K_i} h_k = g_i, \quad \text{for all} \quad i \in I,$$

$$h_k \geq 0, \quad \text{for all} \quad k \in K_i, \quad i \in I;$$

I is the set of origin-destination pairs;
K_i is the set of paths connecting origin-destination pair i;
h_k is the k'th path flow;
g_i is the demand associated with origin-destination pair i.

In developing the algorithm, we consider the following: first, in order to find $H(h)$, one has to solve the quadratic program defined in (11). If $LJBOM$ or $PBOM$ is used, then $B(h)$ or $\overline{B}$ has a block diagonal form and (11) decomposes into a collection of smaller independent quadratic programs, one for each origin-destination pair i; second, since the number of paths in K_i is very large, it is preferable to start with a subset of each K_i and augment this subset as necessary. Bertsekas and Gafni (1982), among others, use a similar restriction method.

We apply Theorem 4.2 with the inexact steplength rule described above and the above two observations to develop the following specific algorithm for the network equilibrium problem.

step 1 (initialization)
Let ϵ be a predetermined tolerance.
Determine initial solutions v^0, $s(v^0)$ and h_k^0, $l = 1$.

step 2 (shortest path computations)
Compute the shortest path k_i based on v^{l-1}
from h^{l-1} for $i \in I$.

step 3 (update path costs)
Compute $S_k(h^{l-1})$, $k \in K_i^{l-1}$, $i \in I$;

step 4 (solve a quadratic program)
Find $H(h^{l-1})$ by solving the projection problems
associated with each origin-destination pair i.

step 5 (determine the descent direction)
$d^{l-1} = H(h^{l-1}) - h^{l-1}$.

step 6 Let $h^l = H(h^{l-1})$ and compute $f(h^{l-1})$.

step 7 (optimality test)
If $\|f(h^{l-1})\| \leq \epsilon$, **then** stop

else continue.

step 8 (linesearch)

step 8.1 update arc volumes: $v^l = \delta h^l$

step 8.2 update arc costs $s(v^l)$

step 8.2 update path costs $S(h^l)$

step 8.3 compute the projection $H(h^l)$

step 8.4 evaluate objective function $f(h^l)$

step 8.5 **If** $f(h^l) \geq f(h^{l-1})$, **then** $h^l = h^{l-1} + \frac{1}{2}(h^l - h^{l-1})$ and goto step 8.1

else $l = l + 1$ and goto step 2.

6 Conclusion.

Motivated by the equivalent optimization formulation for solving monotone variational inequalities found by Fukushima (1989), we extend his results to a more general formulation of VIP_h, which has an application for the network equilibrium problem in the space of path flows. A convergence proof of the descent algorithm is given for VIP_h. This development leads to a new algorithm for the network equilibrium problem.

In the near future, we expect to carry out numerical experiments to verify the effectiveness of the algorithm and compare it to other methods.

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